

Interest Rates

Christopher Ting

<http://www.mysmu.edu/faculty/christophert/>

✉: christopherting@smu.edu.sg

☎: 6828 0364

📍: LKCSB 5036

Table of Contents

- 1 Compounding
- 2 Yield Curve
- 3 Interest Rate Risk
- 4 Yield Curve Shape
- 5 Short-Term Risks
- 6 A Model of Yield Curve

Recap of the First Principle

- ✂ The first principle of Quantitative Finance postulates a frame of reference where risk is absolutely absent.
- ✂ In the risk-free framework, the yield-to-maturity r_0 of the risk-free debt security is called the **risk-free rate**. It connects the present value and the future value when the security matures at time T deterministically.

$$FV = PV \times (1 + r_0)^T. \quad (1)$$

If the present value is 0, the future value remains 0. Otherwise, the value increases steadily as time T increases, which is the **time value of money**.

- ✂ The market practice today is to take the debt securities issued by the U.S. Treasury as a proxy for risk-free instruments.

Compounding

✂ Implicit in (1) is that r_0 is the interest rate per annum and T is the number of years, and thus annual compounding is assumed.

✂ What if compounding occurs every half year? In this case, we have

$$FV = PV \times \left(1 + \frac{r_0}{2}\right)^{2T}.$$


✂ By the same argument, quarterly, monthly, and daily compounding schemes are, respectively,

$$FV = PV \times \left(1 + \frac{r_0}{4}\right)^{4T};$$

$$FV = PV \times \left(1 + \frac{r_0}{12}\right)^{12T};$$


$$FV = PV \times \left(1 + \frac{r_0}{365}\right)^{365T}.$$

Continuous Compounding

-  The first principle of Quantitative Finance is equally applicable to all the compounding schemes, which can be written generically as

$$FV = PV \times \left(1 + \frac{r_0}{n}\right)^{nT},$$

where n is the compounding frequency.

-  What if $n \rightarrow \infty$?

Solution

✂ Note that $x^{nT} = (x^n)^T$, and so consider only the term involving n defined as:

$$y(n) := \left(1 + \frac{r_0}{n}\right)^n.$$

✂ Since the function $y(n)$ is positive for $n \geq 0$, you can take the natural log on both sides of the equation, yielding

$$\ln(y(n)) = n \ln\left(1 + \frac{r_0}{n}\right) = \frac{\ln\left(1 + \frac{r_0}{n}\right)}{\frac{1}{n}}.$$


Solution (Cont'd)

 Note that $\ln(y(n))$ is indeterminate, being of the form $0/0$ as $n \rightarrow \infty$. Hence, **l'Hôpital's rule** applies and we obtain

$$\lim_{n \rightarrow \infty} \ln(y(n)) = \lim_{n \rightarrow \infty} \frac{r_0}{1 + \frac{r_0}{n}} = r_0,$$

which implies that

$$\lim_{n \rightarrow \infty} y(n) = \lim_{n \rightarrow \infty} \left(1 + \frac{r_0}{n}\right)^n = e^{r_0}.$$

 Consequently, when interest is being compounded infinitely many times, or compounded continuously over time, the first principle of Quantitative Finance is written as

$$FV = PV \times e^{r_0 T}.$$

Effective Annual Rate

- ✂ The annual compounding rate is also known as the **simple interest rate**.
- ✂ To compare different compounding schemes, a common practice is to entertain the notion of **effective annual rate** \hat{r} , which is the interest rate that would be obtained if the forward value were to be calculated under the annual compounding scheme.
- ✂ For example, the rate r of continuous compounding is equivalent to \hat{r} via the following equation:

$$FV = PV \times e^{rT} = PV \times (1 + \hat{r})^T.$$

In other words,

$$\hat{r} = e^r - 1.$$

Maturity

- ✂ Another ambiguity that must be disambiguated is the maturity.
- ✂ So far, we have not made any distinction between the risk-free rates of debt securities maturing in 1 month, 3 months (a quarter), 6 months (a half year), one year, and many years.
- ✂ If you plot the risk-free rates against the times to maturity, you will find that the so-called **term structure of interest rates** is a flat line.
- ✂ In other words, the yield-to-maturity curve, or **yield curve** in short, is assumed to be flat with r being the same yield across all maturities at any given point of time t .

Zero-Coupon Bond

- ✂ The yield on the yield curve refers to the return obtained from a **zero-coupon bond**, also known as the **discount bond**.
- ✂ For convenience, let the forward value be fixed at \$1. From (1), the present value becomes the price paid to receive \$1 in the future.

$$P_t = \frac{1}{(1 + r_t)^{T-t}}. \quad (2)$$

- ✂ P_t is the price of a zero-coupon bond at time t . To receive every dollar at the future time T when the bond matures, the price to pay today (present value) $t = 0$ is P_0 .





Long-Term Risk

- ✂ Not all investors hold risk-free securities to maturities.
- ✂ Supply and demand drive the prices of risk-free zero-coupon bonds.
- ✂ Intuitively, compared to the short-term 1-year horizon, long-term investment has more opportunities to see the price of a zero-coupon bond appreciate beyond its purchase price P_0 .
- ✂ Therefore, 10-year or long-term fixed income securities have a higher risk of **opportunity cost** than short-term securities have.
- ✂ Inherently, distant future is uncertain.

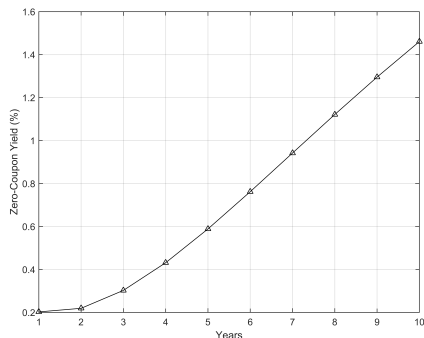
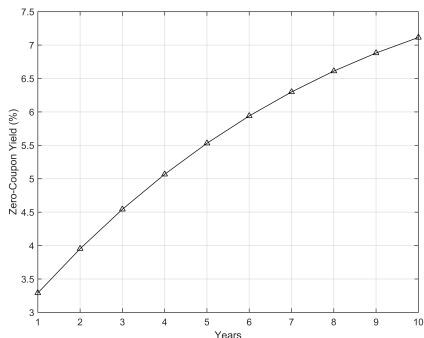
Short-Term Risk

- ✂ Short-term investments are subject to **reinvestment risk**.
- ✂ When short-term securities mature, the resulting proceeds may need to be reinvested at the interest rate on the day of maturity, which is uncertain and could be unfavorable.
- ✂ Compared to 1-year securities, which mature each year to face the reinvestment risk, a 10-year security matures once in 10 years to face the reinvestment risk. In other words, everything else being equal, a 10-year bond has 1/10 reinvestment risk of a 1-year bond.

Quick Summary

-  In sum, the **opportunity cost risk** and the **uncertainty risk** work against the long-term bond whereas the **reinvestment risk** is more pronounced for the short-term security.
-  On balance, opportunity cost risk plus uncertainty risk is usually greater than the reinvestment risk.
-  Given these considerations, the second principle of QF expects the long-term securities to provide a higher expected return or higher yield to compensate investors of long-term bonds for bearing the relatively higher risk. It follows that the yield curve should be **upward sloping** under normal market conditions.
-  Does it mean that there is no such thing as risk-free?
Answer: _____

Examples of U.S. Treasury Yield Curve






Yield Curves of October 7, 1992 (left) and of July 25, 2012

Source: Dataset created and maintained by Gurkaynak, Sack, and Wright (2006)

Concept Checker: Which is the odd one out?

- 1 If a risk-free bond is held to maturity for 20 years, there will be risk because its price changes over time.
- 2 The risk-free rate is constant across maturity because the first principle of QF assumes that investors always hold any risk-free security to maturity.
- 3 The effective interest rate will be larger if compounding is less frequent.
- 4 Even if investors of risk-free securities do not hold them to maturities, the risk-free rate will still be constant according to the first principle of QF.

Monetary Policy Risk

-  The U.S. Treasury yield curve is dependent on the monetary policy of the Federal Reserve, which sets the target for the **federal funds rate** in order to strike a balance between economic growth and price stability.
-  The Federal Reserve buys and sells short-term Treasury securities in the open market so that the market's federal funds rate stays close to the monetary policy's **target federal funds rate**.
-  Short-term Treasury securities are subject to **monetary policy risk**.

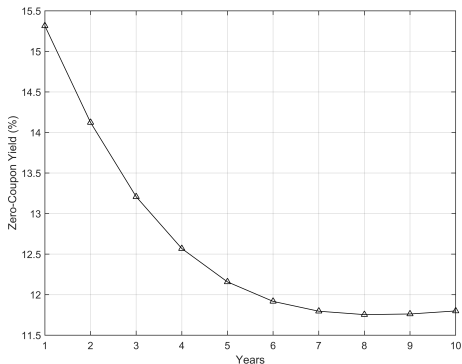
Downward-Sloping Yield Curve



When the short-term risks are a lot riskier, the yield curve may become downward sloping.



In this case, the term structure of interest rates is inverted.



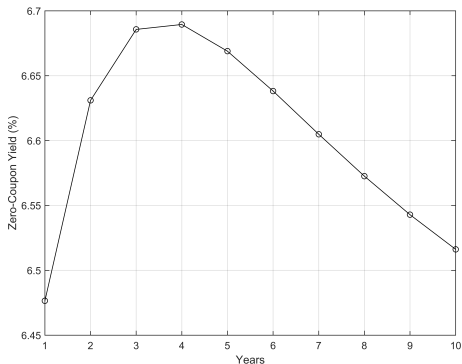
Yield curve of December 12, 1980

Source: Dataset created and maintained by Gurkaynak, Sack, and Wright (2006)

Hump-Shape Yield Curve



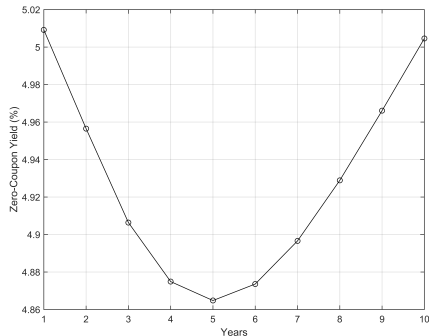
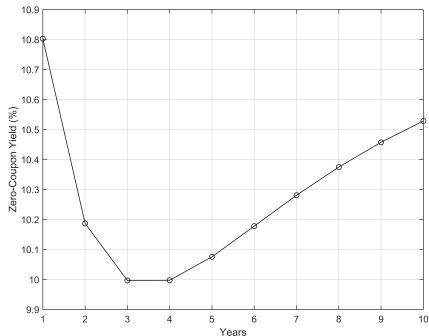
The short-term segment is upward sloping whereas the mid- to long-term segment is downward sloping.



Yield curve of March 5, 1973

Source: Dataset created and maintained by Gurkaynak, Sack, and Wright (2006)

U-Shape Yield Curves



Yield curves of April 29, 1980 (left) and of June 5, 2007





Source: Dataset created and maintained by Gurkaynak, Sack, and Wright

Portfolio of Zero Bonds


- ✂ Term structure of zero rates $\{z_i\}, i = 1, 2, \dots, 2n$.
- ✂ An n -year coupon-bearing is a portfolio of $2n + 1$ zero bonds.
- ✂ The price p for \$1 face value and coupon rate c :

$$p = \frac{c}{2} \sum_{j=1}^{2n} \frac{1}{\left(1 + \frac{z_j}{2}\right)^j} + \frac{1}{\left(1 + \frac{z_{2n}}{2}\right)^{2n}}$$


Bond Price $P(T, t)$

-  An investor buys the bond at a price of $P(T, t)$.
-  Instead of holding the bond to maturity, the investor sells it at a later time $t + \epsilon$.
-  The time to maturity of the bond is no longer T but $T - \epsilon$, and the selling price becomes $P(T - \epsilon, t + \epsilon)$.
-  From the perspective of investment and trading, the bond pricing function $P(T, t)$ reflects the fact that you sell a bond that has a shorter maturity $T - \epsilon$ than you bought earlier at time t .

Bond Return





-  If no coupon is received during the holding period ϵ , the return R for holding the bond over a period of ϵ is

$$R = \frac{P(T - \epsilon, t + \epsilon) - P(T, t)}{P(T, t)}.$$


-  Inserting $0 = -P(T, t + \epsilon) + P(T, t + \epsilon)$ into the numerator gives rise to

$$\begin{aligned} R &= \frac{(P(T - \epsilon, t + \epsilon) - P(T, t + \epsilon)) + (P(T, t + \epsilon) - P(T, t))}{P(T, t)} \\ &= \frac{P(T - \epsilon, t + \epsilon) - P(T, t + \epsilon)}{P(T, t)} + \frac{P(T, t + \epsilon) - P(T, t)}{P(T, t)}. \end{aligned} \tag{3}$$

Bond Return (Cont'd)

-  The first term is the return due to the **passage of time** ϵ in the bond's maturity.
-  The second term captures the return arising from capital appreciation or depreciation of the bond.
-  When the interest rate remains constant, everything else being equal, Treasury bonds of maturity T should sell at the same price, whether at time t or at time $t + \epsilon$.
-  However, if over the period ϵ , interest rates have changed, then $P(T, t + \epsilon)$ will no longer equal $P(T, t)$. This return is thus attributable to the premium for the exposure to the **interest rate risk**.

Return from the Passage of Time

-  The return from the passage of time, denoted by R_d is written as

$$R_d := \frac{P(T - \epsilon, t + \epsilon) - P(T, t + \epsilon)}{P(T, t)} \quad (4)$$

This is the return earned from lending out the money for a period of ϵ .

-  In the numerator, **the Taylor expansion** with respect to maturity T results in

$$R_d = \frac{1}{P} \left(-\frac{\partial P}{\partial T} \epsilon + \frac{1}{2!} \frac{\partial^2 P}{\partial T^2} \epsilon^2 - \frac{1}{3!} \frac{\partial^3 P}{\partial T^3} \epsilon^3 \dots \right. \\ \left. + \dots + (-1)^n \frac{1}{n!} \frac{\partial^n P}{\partial T^n} \epsilon^n + \dots \right). \quad (5)$$

Insight into Risk-Free Interest Rate r_0

Proposition

The return from passage of time R_d is the risk-free rate of the first principle of Quantitative Finance.

Proof

- ✂ Recall that a zero-coupon bond pays \$1 at maturity and the price to pay today is $P = 1/(1 + r_0)^T$, as in (2).
- ✂ If the zero-coupon bond is held to maturity, the (simple) return is

$$R(T) = \frac{1 - \frac{1}{(1 + r_0)^T}}{\frac{1}{(1 + r_0)^T}} = (1 + r_0)^T - 1. \quad (6)$$

Insight into Risk-Free Interest Rate r_0 (Cont'd)



Given that the bond price is explicitly given by

$$P = 1/(1 + r_0)^T = \exp(-T \ln(1 + r_0)),$$

we compute the differentials to obtain

$$\frac{1}{P} \frac{\partial^n P}{\partial T^n} = (-1)^n \ln(1 + r_0)^n, \quad \text{for } n = 1, 2, \dots, \infty.$$



Substituting these differentials into (5), we obtain the return on the discount bond as follows:

$$R_d = \sum_{n=1}^{\infty} \frac{(\ln(1 + r_0)\epsilon)^n}{n!} = \sum_{n=0}^{\infty} \frac{(\ln(1 + r_0)\epsilon)^n}{n!} - 1.$$

Insight into Risk-Free Interest Rate r_0 (Cont'd)

 Hence, by Maclaurin's expansion in the Pre-U math,

$$R_d = \exp(\ln(1 + r_0)\epsilon) - 1 = (1 + r_0)^\epsilon - 1, \quad (7)$$

which is indeed the return $R(T)$ of a discount bond maturing in $\epsilon = T$ years as in (6).

 On the annualized basis, $\epsilon = 1$, and we have

$$R_d = r_0.$$


Q.E.D.

Return due to the Change in Interest Rate

 We define

$$R_s := \lim_{\epsilon \downarrow 0} \frac{P(T, t + \epsilon) - P(T, t)}{P(T, t)} = \frac{dP}{P}.$$


Holding the maturity constant, this return allows you to examine the return in relation to a change in the interest rate.

 To gain further insight, we first define the fluctuation (risk) in interest rate in percentages as follows:

$$\Delta r := \frac{(1 + r + \delta r) - (1 + r)}{1 + r} = \frac{\delta r}{1 + r}. \quad (8)$$

The change in interest rate is denoted by δr in this definition.


Return due to the Change in Interest Rate (Cont'd)

-  Next, we consider the first-order price change with respect to r :

$$dP = \frac{\partial P}{\partial r} \delta r.$$


For the discount bond, the price is $P = 1/(1+r)^T$ as before, and we get

$$\frac{\partial P}{\partial r} = -T \frac{1}{(1+r)^{T+1}} = -T \frac{1}{1+r} P.$$

-  Consequently, the return due to a change in interest rate (yield curve shift) is, up to the first order,


$$R_s = \frac{dP}{P} = -T \frac{\delta r}{1+r} =: -T \Delta r. \quad (9)$$

Discussion




 What does $\Delta r := \frac{\delta r}{1+r}$ mean in financial terms?

 What does the negative sign mean in (9)?

Duration


-  As an example, suppose the discount bond has 10 years to maturity, and the interest rate decreases from 2.25% to 2%. The return is

$$R_s = -10 \times \frac{-0.0025}{1 + 0.0225} = 0.02445.$$


-  If $|\delta r|$ is larger, then the return R_s will be larger.
-  For 1-year bond, the return due to the same interest rate change is 10 times smaller than the 10-year bond.
-  The longer the **duration** D is, the greater will be the sensitivity to the interest rate fluctuation. For this reason, we write generically the price risk as

$$R_s = -D\Delta r. \quad (10)$$

Second-Order Return

-  Up to the second order of the change in interest rate δr , the bond price $P(r + \delta r)$ is given by

$$P(r + \delta r) = P(r) + \frac{\partial P}{\partial r} \delta r + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} (\delta r)^2.$$

-  Accordingly, as a result of interest rate change δr , a more accurate estimate for the bond return is, up to the second order,

$$R_s = \frac{P(r + \delta r) - P(r)}{P(r)} = \frac{1}{P(r)} \left(\frac{\partial P}{\partial r} \delta r + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} (\delta r)^2 \right).$$

-  The percentage price change R_s can thus be written as

$$R_s = -D_m \delta r + \frac{1}{2} C_m (\delta r)^2.$$

Modified Duration and Modified Convexity


-  In view of (10) and the definition of Δr , (8), we define what is known as the **modified duration**:

$$D_m := \frac{D}{1+r}$$

-  The **modified convexity** C_m is defined analogously as

$$C_m := \frac{1}{P} \frac{\partial^2 P}{\partial r^2}.$$


Interest Rate Risk and Return

-  In terms of the percentage change in the interest rate, Δr , the return corresponding to the interest rate risk is

$$R_s = -D\Delta r + \frac{1}{2}C(\Delta r)^2, \quad (11)$$

where the **convexity** C is

$$C := (1 + r)^2 C_m.$$

-  Again we see in (11) that the price risk or interest rate risk is captured in a way that is consistent with the second principle of QF.

A Model of Long-Term Risks




The zero-coupon yields on the yield curve with opportunity cost risk and uncertainty risk taken into consideration can be modeled as

$$Y_T^{(\ell)} = r + T\mathbb{E}(|\Delta r|) + \frac{1}{2}T(T+1)\mathbb{E}((\Delta r)^2). \quad (12)$$




In the context of term structure, r is called the **level** of the yield curve.

Slope of the Yield Curve

 We define a **gross slope** s of the yield curve from the spread between the 10-year and 1-year yields as follows:

$$s = \frac{Y_{10}^{(\ell)} - Y_1^{(\ell)}}{10 - 1} = \mathbb{E}(|\Delta r|) + 6\mathbb{E}((\Delta r)^2). \quad (13)$$

 Since $(\Delta r)^2 = |\Delta r|^2$, the gross slope s is rewritten as

$$s = \mathbb{E}(|\Delta r| + 6|\Delta r|^2). \quad (14)$$

Implied Interest Rate Risk



What is the application of a model?



In QF, oftentimes, a model is used for obtained an **implied quantity**.







An implied absolute percentage change in the interest rate is obtainable from (14) by solving the quadratic equation,

$$|\Delta r|_* = \frac{-1 + \sqrt{1 + 24s}}{12} \approx s. \quad (15)$$






The implied $|\Delta r|_*$ is interpretable as a measure to estimate the gross average interest rate risk across the 1-year to 10-year maturities.


Numerical Example of Implied Interest Rate Risk

-  Consider the upward-sloping curves in Slide 14.
-  The yield curve on October 7, 1992 registers the largest spread ever, i.e., $Y_{10}^{(\ell)} - Y_1^{(\ell)} = 3.8234\%$
-  After divided by 9, the slope $s = 0.4248\%$ per year.
-  The implied $|\Delta r|_*$ works out to be 0.4145% per year, which provides an estimate for the interest rate risk on average across the 1- to 10-year maturities.

Reinvestment Risk Revisited


-  Recall that the reinvestment risk is the risk of facing a potentially lower interest rate at maturity. From the reinvestment perspective, the lower interest rate means a lower yield.
-  To compensate for this risk, the current interest rate must be *lower* so that come maturity, having collected the par value of \$1, the interest rate might be *higher* for reinvestment.
-  The reinvestment risk is negatively related to the time to maturity T .

A Model of Short-Term Risks

-  Inspired by Nelson and Siegel, we postulate that the reinvestment risk is captured by the following function of T :

$$R_r(T) = \frac{1 - e^{-T/\tau}}{\frac{T}{\tau}} - e^{-T/\tau},$$


where τ is a positive parameter that serves as a scale of time.

-  The function $R_r(T)$ can be written as


$$\begin{aligned} R_r(T) &= \frac{e^{-T/\tau} (e^{T/\tau} - 1 - T/\tau)}{\frac{T}{\tau}} \\ &= \frac{e^{-T/\tau} \left(\frac{1}{2}(T/\tau)^2 + \frac{1}{6}(T/\tau)^3 + \dots \right)}{\frac{T}{\tau}}, \end{aligned}$$

which is strictly positive, i.e., $R_r(T) > 0$ for all $T \geq 0$.


A Model of Short-Term Risks (Cont'd)

-  In the yield curve setting, since the yield must be lower as discussed earlier to account for the reinvestment risk, we write

$$Y_T^{(s)} = r - \beta_s \left(\frac{1 - e^{-T/\tau}}{\frac{T}{\tau}} - e^{-T/\tau} \right).$$

-  The parameter β_s acts as the short-term risk premium. It is noteworthy that this model satisfies the important condition that $Y_0 = r$ by applying l'Hôpital's rule as follows:

$$\lim_{T \rightarrow 0} \frac{1 - e^{-T/\tau}}{\frac{T}{\tau}} = 1.$$

-  Consequently, $\lim_{T \rightarrow 0} Y_T^{(s)} = r$, the level of the yield curve!

Modeling the Yield Curve

- ④ In sum, using the symbol Y_T for the T -year zero-coupon yield, we write

$$Y_T = r + \beta^{(\ell)} \left(aT + \frac{b}{2}T(T+1) \right) - \beta^{(s)} \left(\frac{1 - e^{-T/\tau}}{\frac{T}{\tau}} - e^{-T/\tau} \right),$$

where $a = \mathbb{E}(|\Delta r|)$ and $b = \mathbb{E}((\Delta r)^2)$.

- ④ This yield curve model Y_T may be regarded as a **reduced model** of zero-coupon yields.

A Parsimonious Model

$$Y_T = r + \beta^{(l)}T - \beta^{(s)} \left(\frac{1 - e^{-T/\tau}}{\frac{T}{\tau}} - e^{-T/\tau} \right). \quad (16)$$

- As before, r is the level of the yield curve. It will become the risk-free yield if all risks are absent.
- The second parameter is $\beta^{(l)}$, which is the aggregate premium for the long-term risks of opportunity cost and uncertainty. The long-term risks are directly proportional to T , and contribute toward the slope of the yield curve only.
- The nonlinear term in the model contributes toward not only the slope but also the curvature.
- The parameter τ provides a scale of the time T for the short-term risk modeling, which is weighted by the short-term risk premium $\beta^{(s)}$.

Tutorial

- 1 Show that for the parsimonious model, (16), the level of the yield curve is r .
- 2 Show that the gradient of the parsimonious model by differentiation with respect to T is

$$Y'_T = \beta^{(l)} - \beta^{(s)} \left(e^{-T/\tau} \left[\frac{1}{T} + \frac{1}{\tau} \right] - \frac{\tau}{T^2} \left(1 - e^{-T/\tau} \right) \right).$$




- 3 What is the gradient of the parsimonious model when $T \rightarrow 0$?
- 4 What is the gradient of the parsimonious model when $T \rightarrow \infty$?
- 5 Discuss your results for Problems 3 and 4. Intuitively, does it make any sense?

Takeaways

- ➡ To compute interest exactly, you must disambiguate maturity, interest rate, compounding scheme, and day count convention.
- ➡ Term structure of interest rates is usually upward sloping.
- ➡ Return from the passage of time is the return of a zero bond
- ➡ Return due to the change in interest rate is dependent on duration and convexity is a model consistent with the second principle of QF.
- ➡ The shape of the yield curve is characterized by level, slope, and curvature.

Week 4 Assignment from Chapter 4

Exercises

-  Question 1: Compounding scheme is annual
-  Question 2: Compounding scheme is annual
-  Question 4

Additional Exercises

Questions 5, 6